Gravitational waves — Exercise sheet n.1

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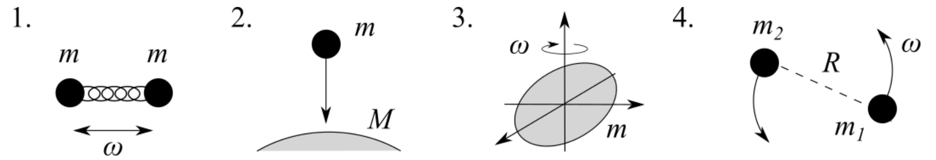
Exercise 1.1: Quadrupole approximation



Consider the following sources of gravitational radiation (see Fig. 1):

1. Two point particles with mass m oscillating with pulsation ω along a fixed axis;
2. Free-falling point-particle with mass m in a Newtonian gravitational field (of mass M);
3. Ellipsoid (with semi-axes a, b, c) rotating around one of its principal axis with frequency ω;
4. Two point particles (with different masses m_1, m_2) in Newtonian circular orbit.

For these cases, compute:

- Inertia tensor of the source,

$$Q^{ij}(t) \equiv I^{ij} - \frac{1}{3}\delta^{ij}\, I^{kk} = \int d^3x\, \rho(t,\vec{x}) \left(x^i x^j - \frac{1}{3} r^2 \delta^{ij} \right) . \qquad (1)$$

  Note that I_{ij} is the standard inertia tensor, while Q_{ij} is the trace-free inertia tensor.

- Gravitational wave emitted in quadrupole approximation in the TT gauge,

$$h_{ij}^{\mathrm{TT}}(t,\vec{x}) = \frac{2G}{r\,c^4}\,\Lambda_{ij,mn}(\theta,\phi)\,\ddot{Q}_{mn}(t - r/c)\,, \qquad (2)$$

  where $\Lambda_{ij,mn}$ is the TT projector.

Exercise 1.2: GW basic scaling

Use the quadrupole formula

$$\bar{h}_{ij} = \frac{2G}{c^4 r} \ddot{I}_{ij}(t-r) \tag{3}$$

to estimate the strain of a source of mass M, extension R, velocity v and at a distance D:

$$h \sim \frac{R}{D}\frac{R}{D}\frac{GM}{c^2 R}\left(\frac{v}{c}\right)^2 . \tag{4}$$

- What is the GW strain from a typical (a) car crash (b) Supernova explosion (c) black hole collision ?
- For each case, What is the relative length displacement that LIGO and Virgo should measure (remember that $h \sim \delta L/L$) ?